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J. Phys. A: Math. Gen. 24 (1991) 1771-1783. Printed in the UK

Kaluza–Klein monopole system in parabolic coordinates by functional integration

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Received 23 October 1990

Abstract. The Kaluza-Klein monopole system is quantized via path integrals in parabolic coordinates. The wavefunctions and energy spectrum of the discrete and continuous spectrum are explicitly evaluated.

1. Introduction

In the framework of quantum mechanics magnetic monopoles have been first discussed by Dirac in his classical paper [1]. He described them as quantized singularities in the electromagnetic field, the quantization condition being

$$2ge = nc\hbar \qquad n \in \mathbb{N} \tag{1}$$

(where e is electric charge, g is magnetic charge and c is velocity of light), arising from the singlevaluedness requirement of the wavefunction. The corresponding Schrödinger equation can be straightforwardly evaluated, see Tamm [2], as well as the propagator (Martinez [3]), leading to a pure continuous spectrum of an electron moving in the field of a magnetic monopole. More general is the Dyon problem, where a Coulomb-interaction term $\alpha eg/r$ is included. This problem has a long history and has been discussed by several authors, see e.g. Barut *et al* [4], Bose [5], Jackiw [6], Schwinger [7], for discussions including spin see D'Hoker and Vinet [8] or the scattering of Dyons (Schwinger *et al* [9]). Of course, the Dyon system has been studied by path integration, where Kleinert [10] had used a Kustaanheimo-Steifel transformation approach, Dürr and Inomata [11] started with a parabolic coordinate formulation and Chetouani *et al* [12] used polar coordinates from the very beginning. In each case a spacetime transformation is needed [13] and the quantization condition (1) arose, however, in various formulations. The much simpler case of the Dirac monopole deserves no spacetime transformation [14].

More elaborated monopole models have been developed since and monopole solutions seem to be inevitable in grand unified theories [15]. Important examples are the Bogomolnyi-Prasad-Sommerfield (BPS) monopoles [16, 17] and Kaluza-Klein monopoles [18, 19], the latter emerging from a static solution of the classical field equations from the former (Taub NUT limit), respectively monopole solutions of five-dimensional Kaluza-Klein gravity [20], where the relevant metric is given by [19]

$$ds^2 = \frac{1}{\Lambda(r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] + \Lambda(r)[4m\, d\psi + A_\phi\, d\phi]^2. \tag{2}$$

Here three-dimensional polar coordinates (r, θ, ϕ) in combination with $x_5 = 8m\psi$ [21] are used and

$$\Lambda(r) = \frac{1}{1+(4m/r)} \qquad A_\phi = 4m(1-\cos\theta). \tag{3}$$

The singularity at the origin vanishes if the coordinate ψ is cyclic with period $16\pi m$ [21]. A thorough study of the classical and quantum properties of the BPS monopoles is due to Gibbons and Manton [17] and they also showed that the Taub NUT limit leeds surprisingly to a Coulomb-like Schrödinger equation which is, of course, exactly solvable. The metric (2) was used by Bernido [21] and Junker and Inomata [22] to establish a path integral solution of this problem in terms of polar coordinates. However, as shown by Gibbons and Manton, this specific monopole problems admits due to its symmetry properties (e.g. Cordani *et al* [18]) also a solution in parabolic coordinates which is quite different from the Dyon problem, being not separable in parabolic coordinates. [Dürr and Inomata [11] started their discussion of the Dyon in parabolic coordinates to establish the quantization condition (1), but later on in their calculation they had to switch back to polar coordinates.]

In this paper, I want to present the Kaluza–Klein monopole problem in a path integral formulation in terms of parabolic coordinates. Whereas the polar coordinate system is more interesting for the study of bound states, the parabolic coordinate system is more suited to the study of scattering problems.

Similar attempts to solve the same problem in different coordinate systems have also been discussed in the Coulomb problem. After the original solution by Duru and Kleinert [13] numerous authors calculated the path integral problem of the hydrogen atom in terms of polar coordinates (see e.g. [23, 24] and references therein) and eventually in parabolic coordinates (Chetouani and Hammann [25]). The motivation thus remains all the time the same, i.e. to achieve as much insight as possible in the system in question and obtain new exact path integral solutions in order to 'build up quantum mechanics from a point of view of fluctuating paths' [13].

The paper is organized as follows: In the next section we set up our notation and describe shortly the construction of path integrals in curved spaces. In section 3 we discuss the Kaluza–Klein monopole system in terms of parabolic coordinates. We start in a specific (radial) coordinate system, and switch after separating the cyclic variable to parabolic coordinates. In our discussion we therefore show that the Kaluza–Klein monopole system is actually separable in parabolic coordinates. Wavefunctions and energy-spectrum for bound and scattering states will be explicitly derived. The section 4 contains a summary and in the appendix the correct normalization of the bound state wavefunctions is shown.

2. Formulation of the path integral

In order to set up our notation we proceed in the canonical way for the formulation of path integrals on curved spaces [23, 26]. We start with the generic case by considering the classical Lagrangian corresponding to the line element $ds^2 = g_{ab}\, dq^a\, dq^b$ of the classical motion in some D-dimensional Riemannian space

$$\mathscr{L}_{\mathrm{Cl}}(q, \dot q) = \frac{m}{2}\left(\frac{ds}{dt}\right)^2 - V(q) = \frac{m}{2} g_{ab}\dot q^a \dot q^b - V(q). \tag{4}$$

The quantum Hamiltonian is *constructed* by means of the Laplace–Beltrami operator

$$H=-\frac{\hbar^2}{2m}\Delta_{\rm LB}+V(q)=-\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^a}g^{ab}\sqrt{g}\frac{\partial}{\partial q^b}+V(q) \tag{5}$$

as a *definition* of the quantum theory on a curved space. Here $g=\det(g_{ab})$ and $(g_{ab})=(g_{ab})^{-1}$. The scalar product for wavefunctions on the manifold reads

$$(f,g)=\int \mathrm{d}q\,\sqrt{g}\,f^*(q)g(q) \tag{6}$$

and the momentum operators which are Hermitian with respect to this scalar product are given by

$$p_a=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial q^a}+\frac{\Gamma_a}{2}\right)\qquad \Gamma_a=\frac{\partial\ln\sqrt{g}}{\partial q^a}. \tag{7}$$

The metric tensor is rewritten as a product according to $g_{ab}=h_{ac}h_{cb}$. Then we obtain for the Hamiltonian (5)

$$H=-\frac{\hbar^2}{2m}\Delta_{\rm LB}+V(q)=\frac{1}{2m}h^{ac}p_ap_bh^{cb}+\Delta V(q)+V(q) \tag{8}$$

the path integral

$$\begin{aligned}
K(q'',q';T)&\\
&=\int \mathscr{D}q(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}h_{ac}h_{cb}\dot q^a\dot q^b-V(q)-\Delta V(q)\right]\mathrm{d}t\right\}\\
&\equiv\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int \mathrm{d}q_{(j)}\sqrt{g(q_{(j)})}\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\varepsilon}h_{bc}(q_{(j)})h_{ac}(q_{(j-1)})\Delta q^{(j),a}\Delta q^{(j),b}\right.\right.\\
&\quad\left.\left.-\varepsilon V(q_{(j)})-\varepsilon\Delta V(q_{(j)})\right]\right\}
\end{aligned} \tag{9}$$

with the well defined quantum potential ΔV given by

$$\begin{aligned}
\Delta V=\frac{\hbar^2}{8m}&[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_b)_{,b}+g^{ab}{}_{,ab}]\\
&+\frac{\hbar^2}{8m}(2h^{ac}h^{bc}{}_{,ab}-h^{ac}{}_{,a}h^{bc}{}_{,b}-h^{ac}{}_{,b}h^{bc}{}_{,a}).
\end{aligned} \tag{10}$$

Here $\Delta q_{(j)}=q_{(j)}-q_{(j-1)}$ for $q_{(j)}=q(t'+j\varepsilon)$ $(\varepsilon=(t''-t')/N=T/N, j=1,\ldots,N)$ with a well defined lattice formulation arising from the ordering prescription for position and momentum operators in the quantum Hamiltonian [27].

3. Kaluza–Klein monopole in parabolic coordinates

In order to describe the Kaluza–Klein monopole system we start (following Inomata and Junker [22]) with the static solution, where the relevant metric is given by (1), (2).

The corresponding Lagrangian of a test particle with mass M consequently has the form

$$\begin{aligned}\mathscr{L}_{\mathrm{Cl}}&=\frac{M}{2}\left\{\frac{1}{\Lambda(r)}[\dot r^2+r^2(\dot\theta^2+\sin^2\theta\dot\phi^2)]+(4m)^2\Lambda(r)[\dot\psi+(1-\cos\theta)\dot\phi]^2\right\}\\&=\frac{M}{2\Lambda(r)}\{\dot r^2+r^2\dot\theta^2+[r^2\sin^2\theta+[4m\Lambda(r)]^2(1-\cos\theta)^2]\dot\phi^2\\&\quad+[4m\Lambda(r)]^2\dot\psi^2+2[4m\Lambda(r)]^2(1-\cos\theta)\dot\phi\dot\psi\}.\end{aligned}\tag{11}$$

Hence the metric has the form [21]

$$(g_{ab})=\frac{1}{\Lambda(r)}\begin{pmatrix}1&0&0&0\\0&r^2&0&0\\0&0&r^2\sin^2\theta+[4m\Lambda(r)]^2(1-\cos\theta)^2&[4m\Lambda(r)]^2(1-\cos\theta)\\0&0&[4m\Lambda(r)]^2(1-\cos\theta)&[4m\Lambda(r)]^2\end{pmatrix}\tag{12}$$

with

$$g=\det(g_{ab})=\left(\frac{4mr^2\sin\theta}{\Lambda(r)}\right)^2\tag{13}$$

and its inverse (g^{ab})

$$(g^{ab})=\Lambda(r)\begin{pmatrix}1&0&0&0\\0&1/r^2&0&0\\0&0&1/r^2\sin^2\theta&-(1-\cos\theta)/(r^2\sin^2\theta)\\0&0&-(1-\cos\theta)/(r^2\sin^2\theta)&1/[4m\Lambda(r)]^2+(1-\cos\theta)^2/(r^2\sin^2\theta)\end{pmatrix}.\tag{14}$$

The canonical momenta

$$\begin{aligned}p_r&=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial r}+\frac{1}{r}-\frac{\Lambda'(r)}{2\Lambda(r)}\right)\\p_\theta&=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta\right)\\p_\phi&=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\phi}\qquad p_\psi=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\psi}\end{aligned}\tag{15}$$

are Hermitian with respect to the scalar product

$$(f,g)=\int\frac{4mr^2\sin\theta}{\Lambda(r)}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi\,f^*(r,\theta,\phi,\psi)g(r,\theta,\phi,\psi)\tag{16}$$

and the quantum potential ΔV has the form

$$\Delta V(r,\theta)=-\frac{\hbar^2\Lambda(r)}{8MR^2}\left(1+\frac{1}{\sin^2\theta}\right).\tag{17}$$

Therefore we have for the quantum Hamiltonian

$$\begin{aligned}H&=-\frac{\hbar^2}{2M}\Lambda(r)\left[\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right.\\&\quad\left.+\left(\frac{1}{[4m\Lambda(r)]^2}+\frac{(1-\cos\theta)^2}{r^2\sin^2\theta}\right)\frac{\partial^2}{\partial\psi^2}-\frac{2}{r^2}\frac{1-\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial\psi}\right]\\&=\frac{1}{2M}h^{ac}p_ap_bh^{cb}-\frac{\hbar^2\Lambda(r)}{8MR^2}\left(1+\frac{1}{\sin^2\theta}\right).\end{aligned}\tag{18}$$

The construction of the path integral is straightforward yielding

$$
\begin{aligned}
K(x'', x'; T) &\\
&\equiv K(r'', r', \theta'', \theta', \phi'', \phi', \psi'', \psi'; T)\\
&= \int \sqrt{g}\, \mathscr{D}r(t)\mathscr{D}\theta(t)\mathscr{D}\phi(t)\mathscr{D}\psi(t)\\
&\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} [\mathscr{L}_{\mathrm{Cl}}(r, \dot r, \theta, \dot\theta, \phi, \dot\phi, \psi, \dot\psi) - \Delta V(r, \theta)]\,\mathrm{d}t\right\}\\
&= \lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\varepsilon\hbar}\right)^{2N}\\
&\quad \times \prod_{j=1}^{N-1}(4m)\int_0^\infty \frac{r_{(j)}^2}{\Lambda(r_{(j)})}\,\mathrm{d}r_{(j)}\int_0^\pi \sin\theta_{(j)}\,\mathrm{d}\theta_{(j)}\int_0^{2\pi}\mathrm{d}\phi_{(j)}\int_0^{16\pi m}\mathrm{d}\psi_{(j)}\\
&\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\varepsilon\Lambda(\widehat{r_{(j)}})}\{\Delta^2 r_{(j)} + \widehat{r_{(j)}^2}\Delta^2\theta_{(j)}\right.\right.\\
&\quad + [\widehat{r_{(j)}^2}\,\widehat{\sin^2}\theta_{(j)} + (4m\Lambda^2(\widehat{r_{(j)}})(1-\widehat{\cos}\,\theta_{(j)})^2]\Delta^2\phi_{(j)}\\
&\quad + 2(4m\Lambda^2(\widehat{r_{(j)}})(1-\widehat{\cos}\,\theta_{(j)})\Delta\phi_{(j)}\Delta\psi_{(j)} + (4m\Lambda^2(\widehat{r_{(j)}})\Delta^2\psi_{(j)}\}\\
&\quad \left.\left. + \frac{\varepsilon\hbar^2\Lambda(r_{(j)})}{8Mr_{(j)}^2}\left[1+\frac{1}{\sin^2\theta_{(j)}}\right]\right)\right\}.
\end{aligned} \tag{19}
$$

Here $f(\widehat{q_{(j)}^2}) := f(q_{(j-1)})f(q_{(j)})$ for any function f of coordinates. In order to evaluate this path integral we start by performing a Fourier expansion in the variable ψ (k denotes some parameter to be more specified later):

$$
\begin{aligned}
K(x'', x'; T) &= \frac{1}{16\pi m}\sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k(\psi''-\psi')}K_k(r'', r', \theta'', \theta', \phi'', \phi'; T)\\
K_k(r'', r', \theta'', \theta', \phi'', \phi'; T) &= \int_0^{16\pi m} K(x'', x'; T)\,\mathrm{e}^{-\mathrm{i}k(\psi''-\psi')}\,\mathrm{d}\psi''.
\end{aligned} \tag{20}
$$

This gives for the kernel $K_k(T)$

$$
\begin{aligned}
K_k(r'', r', \theta'', \theta', \phi'', \phi'; T) &\\
&= \frac{1}{4m[\Lambda(r')\Lambda(r'')]^{1/4}}\lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\varepsilon\hbar}\right)^{3N/2}\\
&\quad \times \prod_{j=1}^{N-1}\int_0^\infty \frac{r_{(j)}^2}{\Lambda^{3/2}(r_{(j)})}\,\mathrm{d}r_{(j)}\int_0^\pi \sin\theta_{(j)}\,\mathrm{d}\theta_{(j)}\int_0^{2\pi}\mathrm{d}\phi_{(j)}\\
&\quad \times \exp\left(\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left\{\frac{m}{2\varepsilon\Lambda(\widehat{r_{(j)}})}[\Delta^2 r_{(j)} + \widehat{r_{(j)}^2}\Delta^2\theta_{(j)}] + k\hbar(1-\widehat{\cos}\,\theta_{(j)})\Delta\phi_{(j)}\right.\right.\\
&\quad \left.\left. -\varepsilon\frac{\hbar^2 k^2}{32m^2 M\Lambda(r_{(j)})} + \frac{\varepsilon\hbar^2\Lambda(r_{(j)})}{8Mr_{(j)}^2}\left(1+\frac{1}{\sin^2\theta_{(j)}}\right)\right\}\right).
\end{aligned} \tag{21}
$$

Therefore, we could perform a separation in terms of monopole harmonics [22]

$$K_k(r'', r', \theta'', \theta', \phi'', \phi'; T) = \frac{1}{2\pi} \sum_{J=|k|}^{\infty} \sum_{M=-J}^{J} (J+\tfrac{1}{2})\, e^{i(M-|k|)(\phi''-\phi')} D^{J*}_{M,k}(\cos\theta') D^{J}_{M,k}(\cos\theta'') K_J(r'', r'; T) \tag{22}$$

with the remaining radial path integral

$$K_J(r'', r'; T) = \frac{[\Lambda(r')\Lambda(r'')]^{1/4}}{4mr'r''} \int \Lambda^{-1/2}(r)\mathscr{D}r(t) \times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2\Lambda(r)}\dot{r}^2 - \hbar^2\Lambda(r)\frac{J(J+1)-k^2}{2mr^2} - \frac{\hbar^2 k^2}{32m^2 M\Lambda(r)}\right] dt\right\} \tag{23}$$

and evaluate the path integral (23) by space-time transformation technique. However, this is not our point here; we instead perform the transformation to parabolic coordinates

$$\begin{aligned} &x = \xi\eta\cos\phi \qquad y = \xi\eta\sin\phi \qquad z = \tfrac{1}{2}(\eta^2 - \xi^2) \qquad \xi, \eta \geqslant 0, \\ &\xi^2 = r + z = r(1+\cos\theta) \qquad \eta^2 = r - z = r(1-\cos\theta) \end{aligned} \tag{24}$$

which gives for the path integral (21)

$$\begin{aligned} K_k(r'', r', \theta'', \theta', \phi'', \phi'; T) &\equiv K_k(\xi'', \xi', \eta'', \eta', \phi'', \phi'; T) \\ &= \frac{1}{4m[\Lambda(r')\Lambda(r'')]^{1/4}} \lim_{N\to\infty}\left(\frac{M}{2\pi i\varepsilon\hbar}\right)^{3N/2} \\ &\times \prod_{j=1}^{N-1}\int_0^\infty d\xi_{(j)} \int_0^\infty d\eta_{(j)} \int_0^{2\pi} d\phi_{(j)} \frac{(\xi^2_{(j)} + \eta^2_{(j)})\xi_{(j)}\eta_{(j)}}{\Lambda^{3/2}(r_{(j)})} \\ &\times \exp\left(\frac{i}{\hbar}\sum_{j=1}^{N}\left\{\frac{m}{2\varepsilon\Lambda(\widehat{r_{(j)}})}[(\widehat{\xi^2_{(j)} + \eta^2_{(j)}})(\Delta^2\xi_{(j)} + \Delta^2\eta_{(j)}) + \widehat{\xi^2_{(j)}}\widehat{\eta^2_{(j)}}\Delta^2\phi_{(j)}]\right.\right. \\ &\left.\left. + k\hbar\Delta\phi_{(j)}\left(1 - \frac{\widehat{\xi^2_{(j)}} - \widehat{\eta^2_{(j)}}}{\widehat{\xi^2_{(j)}} + \widehat{\eta^2_{(j)}}}\right) - \varepsilon\frac{\hbar^2 k^2}{32m^2 M\Lambda(r_{(j)})} + \varepsilon\frac{\hbar^2\Lambda(r_{(j)})}{8M\xi^2_{(j)}\eta^2_{(j)}}\right\}\right). \end{aligned} \tag{25}$$

Note the new quantum potential $\Delta V = -\hbar^2\Lambda(r)/8M\xi^2\eta^2$, $\xi + \eta^2 = 2r$ and we leave for convenience $\Lambda(r)$ as it stands. We perform the time transformation [29]

$$\varepsilon = \frac{\widehat{\xi^2_{(j)} + \eta^2_{(j)}}}{\Lambda(\widehat{r_{(j)}})}\delta = \frac{2\widehat{r_{(j)}}}{\Lambda(\widehat{r_{(j)}})}\delta. \tag{26}$$

This gives the transformation formulae

$$K_k(\xi'', \xi', \eta'', \eta', \phi'', \phi'; T) = \frac{1}{2\pi i\hbar}\int_{-\infty}^{\infty} G_k(\xi'', \xi', \eta'', \eta', \phi'', \phi'; E)\, e^{iET/\hbar}\, dE \tag{27a}$$

$$G_k(\xi'', \xi', \eta'', \eta', \phi'', \phi'; E) = i\int_0^\infty \tilde{K}_k(\xi'', \xi', \eta'', \eta', \phi'', \phi'; s'')\, ds'' \tag{27b}$$

with the path integral $\tilde{K}_k(s'')$ given by

$$\begin{aligned}
&\tilde{K}_k(\xi'', \xi', \eta'', \eta', \phi'', \phi'; s'') \\
&\quad= \frac{[(\xi'^2+\eta'^2)(\xi''^2+\eta''^2)]^{-1/4}}{4m} \lim_{N\to\infty} \left(\frac{M}{2\pi \mathrm{i}\delta\hbar}\right)^{3N/2} \\
&\qquad\times \prod_{j=1}^{N-1} \int_0^\infty \mathrm{d}\xi_{(j)} \int_0^\infty \mathrm{d}\eta_{(j)} \int_0^{2\pi} \mathrm{d}\phi_{(j)} \frac{\xi_{(j)}\eta_{(j)}}{\sqrt{\xi_{(j)}^2+\eta_{(j)}^2}} \\
&\qquad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{M}{2\delta}(\Delta^2\xi_{(j)}+\Delta^2\eta_{(j)}) + \frac{M}{4\delta}\frac{\widehat{\xi_{(j)}^2}\widehat{\eta_{(j)}^2}}{\widehat{r_{(j)}}}\Delta^2\phi_{(j)} + 2\delta E\frac{r_{(j)}}{\Lambda(r_{(j)})}\right.\right. \\
&\qquad\left.\left. + k\hbar\Delta\phi_{(j)}\left(1-\frac{\widehat{\xi_{(j)}^2}-\widehat{\eta_{(j)}^2}}{\widehat{\xi_{(j)}^2}+\widehat{\eta_{(j)}^2}}\right) - \delta\frac{\hbar^2k^2r_{(j)}}{16m^2M\Lambda^2(r_{(j)})} + \frac{\delta\hbar^2 r_{(j)}}{4M\xi_{(j)}^2\eta_{(j)}^2}\right]\right\}.
\end{aligned} \tag{28}$$

We perform a Fourier expansion in the variable ϕ according to

$$\begin{aligned}
\tilde{K}_k(\xi'', \xi', \eta'', \eta', \phi'', \phi'; s'') &= \frac{1}{2\pi}\sum_{\nu=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu(\phi''-\phi')}\tilde{K}_{k\nu}(\xi'', \xi', \eta'', \eta'; s'') \\
\tilde{K}_{k\nu}(\xi'', \xi', \eta'', \eta'; s'') &= \int_0^{2\pi} \mathrm{d}\phi''\, \mathrm{e}^{-\mathrm{i}\mu(\phi''-\phi')}\tilde{K}_k(\xi'', \xi', \eta'', \eta', \phi'', \phi'; s'').
\end{aligned} \tag{29}$$

This gives for the kernel $\tilde{K}_{k\nu}(s'')$

$$\begin{aligned}
&\tilde{K}_{k\nu}(\xi'', \xi', \eta'', \eta'; s'') \\
&\quad= \frac{[(\xi'^2+\eta'^2)(\xi''^2+\eta''^2)]^{-1/4}}{4m} \\
&\qquad\times \lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\delta\hbar}\right)^{3N/2} \prod_{j=1}^{N-1}\int_0^\infty \mathrm{d}\xi_{(j)}\int_0^\infty \mathrm{d}\eta_{(j)} \frac{\xi_{(j)}\eta_{(j)}}{\sqrt{\xi_{(j)}^2+\eta_{(j)}^2}} \\
&\qquad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{M}{2\delta}(\Delta^2\xi_{(j)}+{}^2\eta_{(j)}) + 2\delta E\frac{r_{(j)}}{\Lambda(r_{(j)})}\right.\right. \\
&\qquad\left.\left. -\delta\frac{\hbar^2k^2r_{(j)}}{16m^2M\Lambda^2(r_{(j)})} + \frac{\delta\hbar^2r_{(j)}}{4M\xi_{(j)}^2\eta_{(j)}^2}\right]\right\} \\
&\qquad\times \prod_{j=1}^{N}\int_0^{2\pi}\mathrm{d}\phi_{(j)}\exp\left\{-\frac{M}{4\mathrm{i}\delta\hbar}\frac{\widehat{\xi_{(j)}^2}\widehat{\eta_{(j)}^2}}{\widehat{r_{(j)}}}\Delta^2\phi_{(j)}\right. \\
&\qquad\left. + i\left[k\left(1-\frac{\widehat{\xi_{(j)}^2}-\widehat{\eta_{(j)}^2}}{\widehat{\xi_{(j)}^2}+\widehat{\eta_{(j)}^2}}\right)-\nu\right]\Delta\phi_{(j)}\right\} \\
&\quad= \frac{1}{4m}\exp\left[\frac{8\mathrm{i}ms''}{\hbar}\left(E-\frac{q^2\hbar^2}{M}\right)\right]\frac{1}{\sqrt{\xi'\xi''}}\int\mathcal{D}\xi(s) \\
&\qquad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{M}{2}\dot{\xi}^2-\hbar^2\frac{(\nu-2k)^2-\frac{1}{4}}{2M\xi^2}+\left(E-\frac{\hbar^2q^2}{2M}\right)\xi^2\right]\mathrm{d}s\right\} \\
&\qquad\times\frac{1}{\sqrt{\eta'\eta''}}\int\mathcal{D}\eta(s) \\
&\qquad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{M}{2}\dot{\eta}^2-\hbar^2\frac{v^2-\frac{1}{4}}{2M\eta^2}+\left(E-\frac{\hbar^2q^2}{2M}\right)\eta^2\right]\mathrm{d}s\right\}.
\end{aligned} \tag{30}$$

Here $q=4mk$ and in the $\phi_{(j)}$-integrations we have extended the interval $[0, 2\pi]$ to the entire $\boldsymbol{R}$, which is standard [28]. The path integrals $\tilde{K}_{\nu-2k}(s'')$ and $\tilde{K}_\nu(s'')$ are evaluated as

$$\begin{aligned}
&\tilde{K}_{\nu-2k}(\xi'', \xi'; s'')\\
&\quad=\frac{1}{\sqrt{\xi'\xi''}}\int \mu_{\nu-2k}[\xi^2]\mathscr{D}\xi(s)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{M}{2}\dot{\xi}^2+\left(E-\frac{\hbar^2q^2}{2M}\right)\xi^2\right]\mathrm{d}s\right\}\\
&\quad=\frac{M\omega}{\mathrm{i}\hbar\sin\omega s''}\exp\left[-\frac{M\omega}{2\mathrm{i}\hbar}(\xi'^2+\xi''^2)\cot\omega s''\right]I_{|\nu-2k|}\left(\frac{M\omega\xi'\xi''}{\mathrm{i}\hbar\sin\omega s''}\right)
\end{aligned}\tag{31}$$

$$\begin{aligned}
&\tilde{K}_\nu(\eta'', \eta'; s'')\\
&\quad=\frac{1}{\sqrt{\eta'\eta''}}\int \mathscr{D}\eta(s)\mu_\nu[\eta^2]\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{M}{2}\dot{\eta}^2+\left(E-\frac{\hbar^2q^2}{2M}\right)\eta^2\right]\mathrm{d}s\right\}\\
&\quad=\frac{M\omega}{\mathrm{i}\hbar\sin\omega s''}\exp\left[-\frac{M\omega}{2\mathrm{i}\hbar}(\eta'^2+\eta''^2)\cot\omega s''\right]I_{|\nu|}\left(\frac{M\omega\eta'\eta''}{\mathrm{i}\hbar\sin\omega s''}\right)
\end{aligned}\tag{32}$$

where $\omega^2=(2/M)(\hbar^2q^2/2M-E)$. Here the well-known path integral identity [30, 31]

$$\begin{aligned}
&\int \mathscr{D}r(t)\mu_\nu[r^2]\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot{r}^2-\omega^2r^2)\,\mathrm{d}t\right]\\
&\quad=\frac{m\omega\sqrt{r'r''}}{\mathrm{i}\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2\mathrm{i}\hbar}(r'^2+r''^2)\cot\omega T\right]I_\nu\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right)
\end{aligned}\tag{33}$$

for radial path integrals has been applied with the functional measure $\mu_\nu[r^2]$

$$\begin{aligned}
\mu_\nu[r^2]&=\lim_{N\to\infty}\prod_{j=1}^N \mu_\nu[r_{(j-1)}r_{(j)}]\\
&=\lim_{N\to\infty}\prod_{j=1}^N\left(\frac{2\pi m r_{(j-1)}r_{(j)}}{\mathrm{i}\varepsilon\hbar}\right)^{1/2}\exp\left(-\frac{mr_{(j-1)}r_{(j)}}{\mathrm{i}\varepsilon\hbar}\right)I_\nu\left(\frac{mr_{(j-1)}r_{(j)}}{\mathrm{i}\varepsilon\hbar}\right)
\end{aligned}\tag{34}$$

in order to guarantee a well-defined short-time kernel [23, 31, 32]. I_ν describes a modified Bessel function.

We see that the separation procedure produces a term $i\delta\hbar k^2/Mr$ in (30) which in the case of the non-relativistic Dyon spoils the separability in parabolic coordinates. In the present case this term is cancelled by a term coming from the k^2 summand in the exponent of (28).

To obtain the bound state contribution of the Green function $G_k(E)$ we make use of the Hille–Hardy formula [33, p 1038]:

$$\begin{aligned}
&\frac{t^{-\lambda/2}}{1-t}\exp\left[-\frac{x+y}{2}\frac{1+t}{1-t}\right]I_\lambda\left(\frac{2\sqrt{xyt}}{1-t}\right)\\
&\quad=\sum_{n=0}^\infty\frac{t^nn!}{\Gamma(n+\lambda+1)}(xy)^{\lambda/2}L_n^{(\lambda)}(x)L_n^{(\lambda)}(y)\,\mathrm{e}^{-1/2(x+y)}
\end{aligned}\tag{35}$$

(where $L_n^{(\lambda)}$ are Laguerre polynomials), and get by performing the s'' integration in (27*b*)

$$G_k^{(\mathrm{bound})}(\xi'', \xi', \eta'', \eta', \phi'', \phi'; E)$$
$$= \hbar \sum_{\nu=-\infty}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{\Psi^*_{n_1,n_2,\nu}(\xi', \eta', \phi')\Psi_{n_1,n_2,\nu}(\xi'', \eta'', \phi'')}{E_N - E} \tag{36}$$

with the principal quantum number $N = n_1 + n_2 + \frac{1}{2}|\nu - 2k| + \frac{1}{2}|\nu| + 1$, and the energy levels $(s = 4mq = k)$

$$E_N = \frac{\hbar^2}{(4m)^2 M}\sqrt{N^2 - s^2}(\pm N - \sqrt{N^2 - s^2}) \qquad N \geqslant s. \tag{37}$$

The wavefunctions have the form

$$\Psi_N(\xi, \eta, \phi) \equiv \Psi_{n_1,n_2,\nu}(\xi, \eta, \phi)$$
$$= \frac{\mathrm{e}^{i\nu\phi}}{\sqrt{8\pi m}}\left[\frac{2}{a^3 N^3 \sqrt{N^2 - s^2}} \frac{n_1! n_2!}{\Gamma(n_1 + |\nu - 2k| + 1)\Gamma(n_2 + |\nu| + 1)}\right]^{1/2}$$
$$\times \left(\frac{\xi^2}{aN}\right)^{|\nu-2k|/2} \left(\frac{\eta^2}{aN}\right)^{|\nu|/2} \exp\left(-\frac{\xi^2 + \eta^2}{2aN}\right) L_{n_1}^{(|\nu-2k|)}\left(\frac{\xi^2}{aN}\right) L_{n_2}^{(|\nu|)}\left(\frac{\eta^2}{aN}\right) \tag{38}$$

where $a = 1/(N\sqrt{q^2 - 2E_N M/\hbar^2}) = |4m|/[N(N \mp \sqrt{N^2 - s^2})]$. Let us make some remarks about the quantization condition for k and s, respectively. We consider the interaction term $(1 - \cos\theta)$ in (3). It corresponds in the path integral (21) to a vector potential $\hat{A}_\phi = k\hbar(1 - \cos\theta)/(r \sin\theta)$ so that $(e/c)\hat{A} \cdot \dot{x} = k\hbar(1 - \cos\theta)\dot{\phi}$ which is singular at $\theta = \pi$. $\hat{A}_\phi$ can be changed by a gauge transformation so that $\hat{A} - \check{A} = \nabla X$ with $X = 2k\Delta\phi$ (e.g. [12], also [21]) into a vector potential $\check{A}_\phi = -k\hbar(1 + \cos\theta)/(r \sin\theta)$ which is singular at $\theta = 0$ ($\hat{A}_\phi$ and $\check{A}_\phi$ correspond to the regions *a*) and *b*) of the Wu–Yang potential [34], respectively). The Feynman kernel corresponding to $\check{A}_\phi$ differs from the one of $\hat{A}_\phi$ by a phase factor $\mathrm{e}^{2ik(\phi''-\phi')}$ and the requirement of the single-valuedness of the wavefunctions before and after the transformation thus gives $2s = 2k \in \mathbb{N}$, i.e. a Dirac-like quantization condition is satisfied.

These wavefunctions are correctly normalized (see the appendix). Using the expansion (compare [36, p 158]):

$$\frac{1}{\sin\alpha}\exp[-(x + y)\cot\alpha]I_{2\mu}\left(\frac{2\sqrt{xy}}{\sin\alpha}\right)$$
$$= \frac{1}{2\pi\sqrt{xy}}\int_0^\infty \frac{\Gamma(\frac{1}{2} + \mu + \mathrm{i}p)\Gamma(\frac{1}{2} + \mu - \mathrm{i}p)}{\Gamma^2(1 + 2\mu)}$$
$$\times \mathrm{e}^{-2\alpha p + \pi p} M_{+\mathrm{i}p,\mu}(-2\mathrm{i}x) M_{-\mathrm{i}p,\mu}(+2\mathrm{i}y)\,\mathrm{d}p \tag{39}$$

we obtain the continuous contribution of $G_k(E)$

$$G_k^{(\mathrm{cont})}(\xi'', \xi', \eta'', \eta', \phi'', \phi'; E)$$
$$= \hbar \sum_{\nu=-\infty}^{\infty} \int_0^\infty \mathrm{d}p \int_{-\infty}^{\infty} \mathrm{d}\zeta \frac{\Psi^*_{p,\zeta,\nu}(\xi', \eta', \phi')\Psi_{p,\zeta,\nu}(\xi'', \eta'', \phi'')}{E_p - E} \tag{40}$$

with the energy spectrum

$$E_p = \frac{\hbar^2}{2M}(p^2 + q^2) \tag{41}$$

and the wave-functions

$$\Psi_{p,\zeta,\nu}(\xi,\eta,\phi)$$
$$=\frac{e^{i\nu\phi}}{\sqrt{8\pi m}}\exp\left[\pi|m|\left(p-\frac{q^2}{p}\right)\right]\frac{\Gamma(\frac{1}{2}+|\frac{1}{2}\nu-k|+ip_1)\Gamma((1+|\nu|)/2+ip_2)}{2\pi\xi\eta\Gamma(1+|\nu-2k|)\Gamma(1+|\nu|)}$$
$$\times M_{ip_1,|\nu-2k|/2}(-ip\xi^2)M_{ip_2,|\nu|/2}(-ip\eta^2) \tag{42}$$

where $p_{1,2}=\frac{1}{4}[|4m|(p-q^2/p)\pm 2\zeta]$. The $M_{\mu,\nu}$ denote Whittaker functions.

4. Discussion

In this paper we have studied the Kaluza–Klein monopole system by path integrals in parabolic coordinates. We have found that in this coordinate system the problem is completely separable in contrast to the non-relativistic Dyon system. The energy spectrum of the bound states has the form

$$E_N=\frac{\hbar^2}{(4m)^2M}\sqrt{N^2-s^2}(\pm N-\sqrt{N^2-s^2}) \qquad N\geqslant s. \tag{43}$$

The constraints on bound states are that on the one hand

$$\omega\in\mathbb{R} \qquad \Rightarrow \qquad E<\frac{\hbar^2q^2}{2M} \tag{44}$$

and on the other

$$4m\left(E-\frac{\hbar^2q^2}{M}\right)>0 \qquad \Rightarrow \qquad m<0. \tag{45}$$

This is in accordance with [11, 17]. Gibbons and Manton [17] argued that the energy levels with the '−' signs are artefacts of the asymptotic (Taub NUT) approximation. The levels with the '+' sign give for $N\gg s$

$$E_N\simeq\frac{\hbar^2}{(4m)^2M}\left[\frac{s^2}{2}-\frac{s^2}{8N^2}+O\left(\frac{1}{N^4}\right)\right] \tag{46}$$

which exhibits a Coulomb-like behaviour.

Let us note that we can calculate the entire Green function

$$G_k(\xi'',\xi',\eta'',\eta',\phi'',\phi';E)$$
$$=G_k^{(\mathrm{bound})}(\xi'',\xi',\eta'',\eta',\phi'',\phi';E)+G_k^{(\mathrm{cont})}(\xi'',\xi',\eta'',\eta',\phi'',\phi';E) \tag{47}$$

in closed form by using the 'addition theorem' [35, vol II, p 99]:

$$\frac{z}{2}J_\nu(z\sin\alpha\sin\beta)J_\mu(z\cos\alpha\cos\beta)$$
$$=(\sin\alpha\sin\beta)^\nu(\cos\alpha\cos\beta)^\mu$$
$$\times\sum_{l=0}^{\infty}(-1)^l(\mu+\nu+2l+1)\frac{\Gamma(\mu+\nu+l+1)\Gamma(\nu+l+1)}{l!\Gamma^2(\nu+1)\Gamma(\mu+l+1)}J_{\mu+\nu+2l+1}(z)$$
$$\times{}_2F_1(-l,\mu+\nu+l+1;\nu+1;\sin^2\alpha)$$
$$\times{}_2F_1(-l,\mu+\nu+l+1;\nu+1;\sin^2\beta). \tag{48}$$

This yields (note that it is more convenient to switch back to polar coordinates and we assume without loss of generality $r''\geqslant r'$)

$$
\begin{aligned}
&G_k(\xi'',\xi',\eta'',\eta',\phi'',\phi';E)\\
&\quad=\frac{1}{8\pi m}\sum_{\nu=-\infty}^{\infty}e^{i\nu(\phi''-\phi')}\left(\sin\frac{\theta'}{2}\sin\frac{\theta''}{2}\right)^{\lambda_1}\left(\cos\frac{\theta'}{2}\cos\frac{\theta''}{2}\right)^{\lambda_2}\\
&\qquad\times\sum_{l=0}^{\infty}(\lambda_1+\lambda_2+2l+1)\frac{\Gamma(\lambda_1+\lambda_2+l+1)\Gamma(\lambda_1+l+1)}{l!\,\Gamma^2(\lambda_1+1)\Gamma(\lambda_2+l+1)}\\
&\qquad\times{}_2F_1\left(-l,\lambda_1+\lambda_2+l+1;\lambda_1+1;\sin^2\frac{\theta'}{2}\right)\\
&\qquad\times{}_2F_1\left(-l,\lambda_1+\lambda_2+l+1;\lambda_1+1;\sin^2\frac{\theta''}{2}\right)\\
&\qquad\times\frac{M\omega}{\sqrt{r'r''}\,\hbar}\int_0^{\infty}\frac{ds''}{\sin\omega s''}\exp\left[\frac{8imS''}{\hbar}\left(E-\frac{q^2\hbar^2}{M}\right)\right]\\
&\qquad\times\exp\left[-\frac{m\omega}{i\hbar}(r'^2+r''^2)\cot\omega s''\right]I_{\lambda_1+\lambda_2+2l+1}\left(\frac{2M\omega\sqrt{r'r''}}{i\hbar\sin\omega s''}\right)\\
&\quad=\frac{1}{8\pi m}\sum_{\nu=-\infty}^{\infty}e^{i\nu(\phi''-\phi')}\left(\sin\frac{\theta'}{2}\sin\frac{\theta''}{2}\right)^{\lambda_1}\left(\cos\frac{\theta'}{2}\cos\frac{\theta''}{2}\right)^{\lambda_2}\\
&\qquad\times\sum_{l=0}^{\infty}\frac{\lambda_1+\lambda_2+2l+1}{2}\,\frac{\Gamma(\lambda_1+\lambda_2+l+1)l!}{\Gamma(\lambda_1+l+1)\Gamma(\lambda_2+l+1)}\\
&\qquad\times P_l^{(\lambda_1,\lambda_2)}(\cos\theta')P_l^{(\lambda_1,\lambda_2)}(\cos\theta'')\\
&\qquad\times\frac{1}{\omega r'r''}\,\frac{\Gamma[\frac{1}{2}(\lambda_1+\lambda_2+2l+2)-a/\hbar\omega]}{\Gamma(\lambda_1+\lambda+2+2l+2)}\\
&\qquad\times W_{a/\hbar\omega,l+(1+\lambda_1+\lambda_2)/2}\left(\frac{2M\omega}{\hbar}r''\right)M_{a/\hbar\omega,l+(1+\lambda_1+\lambda_2)/2}\left(\frac{2M\omega}{\hbar}r'\right)
\end{aligned}\tag{49}
$$

($\lambda_1=|\nu-2k|$, $\lambda_2=|\nu|$, $a=4m[E-(q^2\hbar^2/M)]$), ω as in (31, 32) and the $P_n^{(a,b)}(x)$ denote Jacobi polynomials. This is basically the result of [21, 22]. Here we have used the integral representation [33, p 1059]

$$
\begin{aligned}
&\int_0^{\infty}\left(\coth\frac{x}{2}\right)^{2\nu}\exp\left(-\frac{a+b}{2}\cosh x\right)I_{2\mu}(\sqrt{ab}\sinh x)\,dx\\
&\quad=\frac{\Gamma(\frac{1}{2}+\mu-\nu)}{\sqrt{ab}\,\Gamma(1+2\mu)}W_{\nu,\mu}(a)M_{\nu,\mu}(a)
\end{aligned}\tag{50}
$$

(where $M_{\nu,\mu}$, $W_{\nu,\mu}$ are Whittaker functions), performed in the s'' integration the substitution $u=-i\omega s''$, together with a Wick rotation, followed by a second substitution $\sinh x=1/\sinh u$. Thus we have established an independent solution of the Kaluza-Klein monopole path integral problem in an alternative coordinate system.

Acknowledgment

This work was supported by Deutsche Forschungsgemeinschaft under contract no. DFG Gr 1031.

Appendix. Check of the normalization of the bound state wavefunctions

Let us check the normalization of the bound state wavefunctions (38). We have

$$\|\Psi_N\|^2=\frac{2}{a^3N^3\sqrt{N^2-s^2}}\frac{n_1!\,n_2!}{\Gamma(n_1+|\nu-2k|+1)\Gamma(n_2+|\nu|+1)}$$
$$\times\int\xi\eta\,\mathrm{d}\xi\,\mathrm{d}\eta\frac{\xi^2+\eta^2}{\Lambda(r)}\left(\frac{\xi^2}{aN}\right)^{|\nu-2k|}\left(\frac{\eta^2}{aN}\right)^{|\nu|}$$
$$\times\exp\left(-\frac{\xi^2+\eta^2}{aN}\right)\left[L_{n_1}^{(|\nu-2k|)}\left(\frac{\xi^2}{aN}\right)L_{n_2}^{(|\nu|)}\left(\frac{\eta^2}{aN}\right)\right]^2$$
$$=\frac{1}{2\sqrt{N^2-s^2}}\frac{n_1!\,n_2!}{\Gamma(n_1+|\nu-2k|+1)\Gamma(n_2+|\nu|+1)}$$
$$\times\left\{\frac{8m}{aN}\int_0^\infty\mathrm{d}x\,x^{|\nu-2k|}\,\mathrm{e}^{-x}[L_{n_1}^{(|\nu-2k|)}(x)]^2\int_0^\infty\mathrm{d}y\,y^{|\nu|}\,\mathrm{e}^{-y}[L_{n_1}^{|\nu|)}(y)]^2\right.$$
$$+\int_0^\infty\mathrm{d}x\,x^{|\nu-2k|+1}\,\mathrm{e}^{-x}[L_{n_1}^{(|\nu-2k|)}(x)]^2\int_0^\infty\mathrm{d}y\,y^{|\nu|}\,\mathrm{e}^{-y}[L_{n_1}^{(|\nu|)}(y)]^2$$
$$\left.+\int_0^\infty\mathrm{d}x\,x^{|\nu-2k|}\,\mathrm{e}^{-x}[L_{n_1}^{(|\nu-2k|)}(x)]^2\int_0^\infty\mathrm{d}y\,y^{|\nu|+1}\,\mathrm{e}^{-y}[L_{n_1}^{(|\nu|)}(y)]^2\right\}$$
$$=\frac{1}{\sqrt{N^2-s^2}}\left[\frac{4m}{aN}+\left(n_1+n_2+\frac{|\nu-2k|}{2}+\frac{|\nu|}{2}+1\right)\right]$$
$$=\frac{N}{\sqrt{N^2-s^2}}\left(1+\frac{4m}{aN^2}\right)=1.$$

Here use has been made of the integral formula [36, p 143]

$$\int_0^\infty\exp\left[-t\left(s+\frac{\alpha_1+\alpha_2}{2}\right)\right]t^{\mu+\beta}L_\lambda^{(\mu)}(\alpha_1t)L_\lambda^{(\mu)}(\alpha_2t)\,\mathrm{d}t$$
$$=(-1)^\lambda\frac{\Gamma(1+\mu+\lambda)}{\lambda!a_0^{1+\mu+\beta}}\sum_{n=0}^{\lambda}\left(-\frac{a_2}{a_0}\right)^n\binom{\beta}{\lambda-n}$$
$$\times\frac{\Gamma(1+\mu+\beta+n)}{\Gamma(1+\mu+n)}P_n^{(\mu,\beta)}\left(\frac{a_1^2}{a_0a_2}\right)$$

($a_0=s+(\alpha_1+\alpha_2)/2$, $a_2=s-(\alpha_1+\alpha_2)/2$, $a_1=a_0a_2+2\alpha_1\alpha_2$) and I have explicitly inserted $a=|4m|/[N(N-\sqrt{N^2-s^2})]$. Note that only for the '+' sign in (43) we have $\|\Psi_N\|>0$. Thus the wavefunctions $\Psi_N(\xi,\eta,\phi)$ have the correct normalization.

References

[1] Dirac P A M 1931 *Proc. R. Soc.* A **133** 60
[2] Tamm Ig 1931 *Z. Phys.* **71** 141
[3] Martinez J C 1987 *J. Phys. A: Math. Gen.* **20** L61

[4] Barut A O, Schneider C K E and Wilson R 1979 *J. Math. Phys.* **20** 2244
[5] Bose S K 1985 *J. Phys. A: Math. Gen.* **18** 1289
[6] Jackiw R 1980 *Ann. Phys., NY* **129** 183
[7] Schwinger 1969 *Science* **165** 757
[8] D'Hoker E and Vinet L 1985 *Nucl. Phys.* B **260** 79
[9] Schwinger J, Milton K A, Tsai W-Y, DeRaad L L Jr and Clark D C 1976 *Ann. Phys., NY* **101** 451
[10] Kleinert H 1986 *Phys. Lett.* **116A** 201
[11] Dürr H and Inomata A 1985 *J. Math. Phys.* **26** 2231
[12] Chetouani L, Guechi L, Letlout M and Hammann T F 1990 *Nuovo Cimento* B **105** 387
[13] Duru I H and Kleinert H 1979 *Phys. Lett.* **84B** 185; 1982 *Fort. Phys.* **30** 401
[14] Grosche C 1990 *Phys. Lett.* A **151** 365
[15] Craigie, N S, Goddard P and Nahm W (ed) 1982 *Monopoles in Quantum Field Theory* (Singapore: World Scientific).
[16] Polyakov A M 1974 *JETP Lett.* **20** 194
t'Hooft G 1974 *Nucl. Phys.* B **79** 276
Prasad M K and Sommerfield C M 1975 *Phys. Rev. Lett.* **35** 760
Bogomol'nyi E B 1976 *Sov. J. Nucl. Phys.* **24** 449
Bogomol'nyi E B and Marinov M S 1976 *Sov. J. Nucl. Phys.* **23** 355
[17] Gibbons G W and Manton N S 1986 *Nucl. Phys.* B **274** 183
[18] Cordani B, Fehér Gy and Horvàthy P A 1988 *Phys. Lett.* B **201** 481
[19] Salam A and Strathdee J 1982 *Ann. Phys., NY* **141** 316
Gross D J and Perry M J 1983 *Nucl. Phys.* B **226** 29
Rafael D Sorkin 1983 *Phys. Rev. Lett.* **51** 87
[20] Kaluza Th 1921 *Sitzungsberichte. Preuss. Akad. Wiss. Math. Phys.* 895
Klein O 1926 *Z. Phys.* **37** 895
[21] Bernido C C 1987 *Phys. Lett.* **125A** 176; 1989 *Nucl. Phys.* B **321** 108; 1989 *Path Integrals From meV to MeV* ed V Sa-yakanit *et al* (Singapore: World Scientific, p 238)
[22] Inomata A and Junker G 1990 *Phys. Lett.* **234A** 41
[23] Grosche C and Steiner F 1987 *Z. Phys.* C **36** 699
[24] Chetouani L and Hammann T F 1986 *J. Math. Phys.* **27** 2944
[25] Chetouani L and Hammann T F 1986 *Phys. Rev.* A **34** 4737; 1987 *Nuovo Cimento* B **98** 1; 1987 *Phys. Lett.* **125A** 277
[26] De Witt B S 1957 *Rev. Mod. Phys.* **29** 377
McLaughlin D W and Schulman L S 1971 *J. Math. Phys.* **12** 2520
Mayes I M and Dowker J S 1972 *Proc. R. Soc.* A **327** 131
Mizrahi M M 1975 *J. Math. Phys.* **16** 2201
Gervais J-L and Jevicki A 1976 *Nucl. Phys.* B **110** 93
Omote M 1977 *Nucl. Phys.* B **120** 325
Marinov M S 1980 *Phys. Rep.* **60** 1
Lee T D 1988 *Particle Physics and Introduction to Field Theory* (Harwood: Academic)
D'Olivio J C and Torres M 1988 *J. Phys. A: Math. Gen.* **21** 3355; 1988 *Path Summation: Achievements and Goals, Trieste 1987* ed S Lundquist, A Ranfagni, V Sa-yakanit and L S Schulman (Singapore: World Scientific) p 469
[27] Grosche C 1988 *Phys. Lett.* **128A** 113
[28] Arthurs A M 1969 *Proc. R. Soc.* A **313** 445
[29] Kleinert H 1987 *Phys. Lett.* **120A** 361
[30] Peak D and Inomata A 1969 *J. Math. Phys.* **10** 1422
Duru I H 1985 *Phys. Lett.* **112A** 421
[31] Steiner F 1985 *Bielefeld Encounters in Physics and Mathematics VII: Path Integrals From MeV to MeV* ed M C Gutzwiller *et al* (Singapore: World Scientific) p 335
[32] Kleinert 1989 *Phys. Lett.* **224B** 313
[33] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic)
[34] Wu T T and Yang C N 1976 *Nucl. Phys.* B **107** 365; 1977 *Phys. Rev.* **16** 1018
[35] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1985 *Higher Transcendental Functions* (New York: McGraw-Hill)
[36] Buchholz H 1969 *The Confluent Hypergeometric Function* (*Springer Tracts in Natural Philosophy* **15**) (Berlin: Springer)